Factoraial Experiments and Interaction Effects

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Factorial Design

- Effects of multiple treatments
  - Example: candidate choice
    1. Partisanship
    2. Race
    3. Gender
    4. Policy
    5. Education
    6. $\vdots$

- Simplest case: 2×2 factorial design
  - Units: $i = 1, \ldots, n$
  - Two treatments: $T_{1i} \in \{0, 1\}$ and $T_{2i} \in \{0, 1\}$
  - Potential outcomes: $(Y_i(0,0), Y_i(1,0), Y_i(0,1), Y_i(1,1))$
  - Independent, complete randomization of T_{1i} and T_{2i}: for all i,
    $$(Y_i(0,0), Y_i(1,0), Y_i(0,1), Y_i(1,1)) \perp\!\!\!\perp (T_{1i}, T_{2i}) \quad \text{and} \quad T_{1i} \perp\!\!\!\perp T_{2i}$$
  - Random sampling of units:
    $$(Y_i(0,0), Y_i(1,0), Y_i(0,1), Y_i(1,1)) \overset{\text{i.i.d.}}{\sim} \mathcal{F}$$

Higher-order Example: Conjoint Design

Please read the descriptions of the potential immigrants carefully. Then, please indicate which of the two immigrants you would personally prefer to see admitted to the United States.

	Immigrant 1	Immigrant 2
Prior Trips to the U.S.	Entered the U.S. once before on a tourist visa	Entered the U.S. once before on a tourist visa
Reason for Application	Reunite with family members already in U.S.	Reunite with family members already in U.S.
Country of Origin	Mexico	Iraq
Language Skills	During admission interview, this applicant spoke fluent English	During admission interview, this applicant spoke fluent English
Profession	Child care provider	Teacher
Job Experience	One to two years of job training and experience	Three to five years of job training and experience
Employment Plans	Does not have a contract with a U.S. employer but has done job interviews	Will look for work after arriving in the U.S.
Education Level	Equivalent to completing two years of college in the U.S.	Equivalent to completing a college degree in the U.S.
Gender	Female	Male

Hainmueller et. al. (2014)

Combination Effect

- Population average combination effect (ACE):
$$\tau\left(\mathbf{t};\mathbf{t}'\right) \equiv \mathbb{E}\left[Y_i\left(t_1,t_2\right) - Y_i\left(t'_1,t'_2\right)\right]$$
where
$$\mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \mathbf{t}' = \begin{pmatrix} t'_1 \\ t'_2 \end{pmatrix}$$
and $t_1, t_2, t'_1, t'_2 \in \{0, 1\}$
- Does **NOT** assume
$$\tau\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix};\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \tau\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix};\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) + \tau\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix};\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$
- Difference-in-means estimator:
$$\widehat{\tau\left(\mathbf{t};\mathbf{t}'\right)} = \frac{1}{n_{\mathbf{t}}}\sum_{i=1}^{n} Y_i 1\left\{T_{1i} = t_1, T_{2i} = t_2\right\} - \frac{1}{n_{\mathbf{t}'}}\sum_{i=1}^{n} Y_i 1\left\{T_{1i} = t'_1, T_{2i} = t'_2\right\}$$
where $1\{\cdot\}$ is the indicator function and $n_{\mathbf{t}}$ is the number of observations with treatment combination $\mathbf{t}$
  - As the number of combinations $\uparrow \implies n_{\mathbf{t}}$ and $n_{\mathbf{t}'} \downarrow$
  - All results (unbiasedness, variance estimation, regression, etc.) for diff-in-means in standard experiments hold

Marginal Effect

- May be interested in the effects of the treatments separately
- Population average marginal effect (AME) of $T_k, k = 1, 2$:

$$\tau_k \equiv \sum_{t=0}^{1} \mathbb{E}\left[\left(Y_i \left(T_{ki} = 1, T_{k'i} = t \right) - Y_i \left(T_{ki} = 0, T_{k'i} = t \right) \right) \right] \mathbb{P}\left(T_{k'i} = t \right)$$

  - E.g., $\tau_1 = \mathbb{E}\left[\sum_{t=0}^{1} \left(Y_i(1, t) - Y_i(0, t) \right) \mathbb{P}\left(T_{2i} = t \right) \right]$
  - Weighted average of ACEs; weights are the probabilities of T_{2i}
  - *Estimand depends on the design*
- Difference-in-means estimator:

$$\hat{\tau}_k = \frac{1}{n_{k1}} \sum_{i=1}^{n} Y_i 1\left\{ T_{ki} = 1 \right\} - \frac{1}{n_{k0}} \sum_{i=1}^{n} Y_i 1\left\{ T_{ki} = 0 \right\}$$

where $n_{kt}, t = 0, 1$ is the number of units with $T_{ki} = t$

  - To prove unbiasedness, use:

$$1\left\{ T_{ki} = t \right\} = 1\left\{ T_{ki} = t \right\} \sum_{t'=0}^{1} 1\left\{ T_{k'i} = t' \right\}$$

$$\mathbb{E}\left[1\left\{ T_{k'i} = t' \right\} \right] = \mathbb{P}\left(T_{k'i} = t' \right)$$

Linear Regression for AME

- Can we use linear regression?
- Consider the population regression of Y_i on T_{1i}:

$$(\alpha, \beta) \equiv \operatorname*{argmin}_{a,b} \mathbb{E}\left[\left(Y_i - a - bT_{1i}\right)^2\right]$$

  - T_{1i} is binary, so we know $\alpha + \beta T_{1i} = \mathbb{E}[Y_i \mid T_{1i}]$
  - Then,

$$\begin{aligned} \alpha + \beta t =& \mathbb{E}[Y_i \mid T_{1i} = t] \\ =& \mathbb{E}[Y_i \mid T_{1i} = t, T_{2i} = 0]\mathbb{P}(T_{2i} = 0) \\ & + \mathbb{E}[Y_i \mid T_{1i} = t, T_{2i} = 1]\mathbb{P}(T_{2i} = 1) \quad \because T_{1i} \perp\!\!\!\perp T_{2i} \\ =& \mathbb{E}[Y_i(t, 0)]\mathbb{P}(T_{2i} = 0) \\ & + \mathbb{E}[Y_i(t, 1)]\mathbb{P}(T_{2i} = 1) \\ & \because (Y_i(0,0), Y_i(1,0), Y_i(0,1), Y_i(1,1)) \perp\!\!\!\perp (T_{1i}, T_{2i}) \end{aligned}$$

  - Therefore, $\beta = \tau_1$!
- Estimand of simple linear regression = AME
  - Key is $T_{1i} \perp\!\!\!\perp T_{2i}$ (will come back later)
  - Robust variance estimation recommended

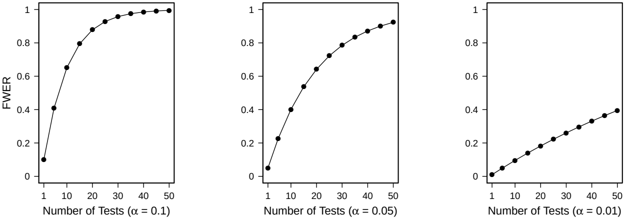
Multiple Hypothesis Testing

- Two types of errors statistical tests can make:

	Reject H_0	Retain H_0
H_0 is true	Type I error	Correct
H_0 is false	Correct	Type II error

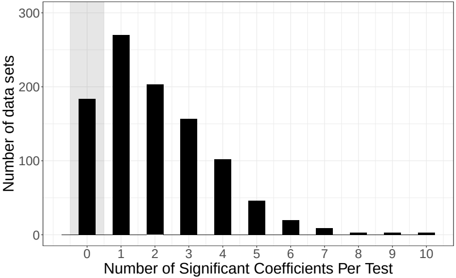
- Test one hypothesis, $\alpha \equiv \mathbb{P}(\text{Reject null} \mid \text{Null is true}) = 0.05$
- Family-Wise Error Rate $\equiv \mathbb{P}(\text{At least one true null is rejected})$
- Test 10 true null hypotheses simultaneously with $\alpha = 0.05$

$$\text{FWER} = 1 - (1 - \alpha)^{10} \approx .4$$

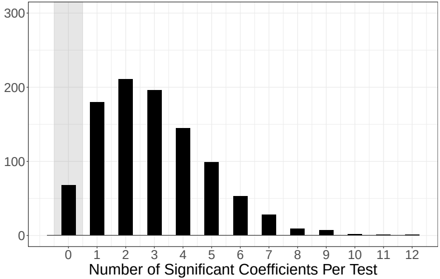


Quantifying the Problem by Simulations

- If AMCE is zero, in how many samples do you get false findings?
- Two scenarios for 41 attribute levels:
  1. No *individual* effect
  2. Nonzero individual effect, but zero *average* effect
- Number of samples for each number of false findings:



Zero individual MCE



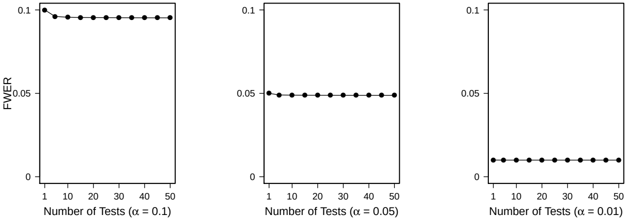
Heterogeneous MCE, zero AMCE

Liu and Shiraito (2023)

Bonferroni Correction

- Setup
  - m: # of tests
  - $p_j, j = 1, \ldots, m$: p-value obtained for test j
  - m_0: # of true null hypotheses (unknown)
  - m_1: # of false null hypotheses (unknown), $m_0 + m_1 = m$
- Bonferroni correction: for each j, reject null if $p_j \leq a/m$
- FWER is controlled:

$$\mathbb{P}\left(\bigcup_{\{j:\text{true null}\}} p_j \leq \frac{a}{m}\right) \leq \sum_{\{j:\text{true null}\}} \mathbb{P}\left(p_j \leq \frac{a}{m}\right) = m_0 \frac{a}{m} \leq a$$



Benjamini-Hochberg Procedure

- Number of discoveries

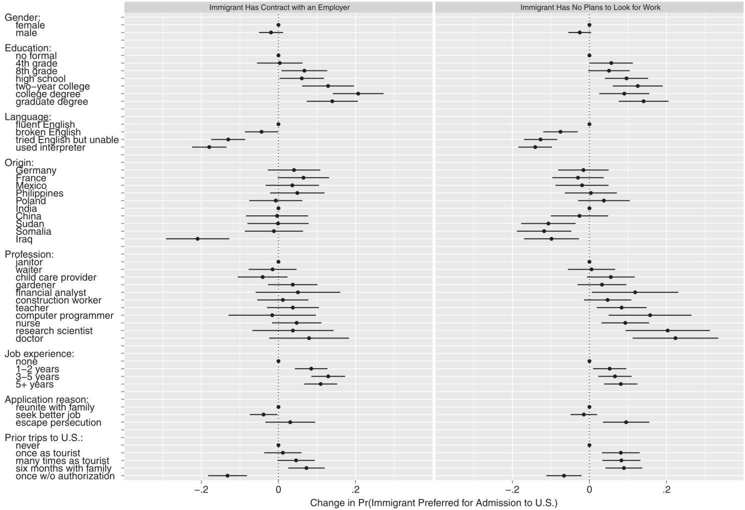
	Reject H_0	Retain H_0	Total
H_0 is true	V	U	m_0
H_0 is false	S	T	m_1
Total	R	$m - R$	m

- False Discovery Rate $\equiv \mathbb{E}[V/R]$ and 0 if $R = 0$
  - If $m_0 = m$, FDR $=$ FWER
  - If $m_0 < m$, FDR $\leq$ FWER $\rightsquigarrow$ controling FDR is less stringent
- Benjamini-Hochberg procedure:
  1. set desired FDR q
  2. Reorder p-values from smallest to largest, $p_{(1)} \leq \cdots \leq p_{(m)}$
  3. Find the largest i such that $p_{(i)} \leq iq/m$
  4. Reject null in tests $(1), \ldots, (i)$
- Example: p-values $(.012, .013, .016, .023, .033, .047, .33, .44)$
  1. let $q = .05$
  2. $.023$ is the largest: $.023 \leq 4 \times .05/8$
  3. reject 4 nulls, instead of 6

Interaction Effects

- When there multiple treatments, they may *interact*
- Population average interaction effect (AIE):
$$\xi(\mathbf{t};\mathbf{t}') \equiv \mathbb{E}\left[Y_i(t_1,t_2) - Y_i(t_1',t_2) - Y_i(t_1,t_2') + Y_i(t_1',t_2')\right]$$
- Interactive effect interpretation:
$$\xi(\mathbf{t};\mathbf{t}') = \underbrace{\mathbb{E}\left[Y_i(t_1,t_2) - Y_i(t_1',t_2')\right]}_{\tau(\mathbf{t};\mathbf{t}')} - \underbrace{\left(\mathbb{E}\left[Y_i(t_1',t_2) - Y_i(t_1',t_2')\right] + \mathbb{E}\left[Y_i(t_1,t_2') - Y_i(t_1',t_2')\right]\right)}_{\text{sum of the effect of each treatment}}$$
  - "Additional effect of combining the two treatments"
- Conditional effect interpretation:
$$\xi(\mathbf{t};\mathbf{t}') = \underbrace{\mathbb{E}\left[Y_i(t_1,t_2) - Y_i(t_1,t_2')\right]}_{\text{PATE of } T_2 \text{ when } T_1=t_1} - \underbrace{\mathbb{E}\left[Y_i(t_1',t_2) - Y_i(t_1',t_2')\right]}_{\text{PATE of } T_2 \text{ when } T_1=t_1'}$$
$$= \underbrace{\mathbb{E}\left[Y_i(t_1,t_2) - Y_i(t_1',t_2)\right]}_{\text{PATE of } T_1 \text{ when } T_2=t_2} - \underbrace{\mathbb{E}\left[Y_i(t_1,t_2') - Y_i(t_1',t_2')\right]}_{\text{PATE of } T_1 \text{ when } T_2=t_2'}$$
  - "How does the effect of a treatment depend on the other?"

Conditional Effects



Hainmueller et. al. (2014)

Linear Regression and Interaction

- "Saturated" model: indicator variable for each treatment combination
$$\begin{aligned} Y_i = &\delta_{00} 1\{T_{1i} = 0, T_{2i} = 0\} + \delta_{01} 1\{T_{1i} = 0, T_{2i} = 1\} \\ &+ \delta_{10} 1\{T_{1i} = 1, T_{2i} = 0\} + \delta_{11} 1\{T_{1i} = 1, T_{2i} = 1\} + \varepsilon_i \end{aligned}$$
- Saturated model = conditional expectation function
$$\mathbb{E}[Y_i \mid T_{1i} = 0, T_{2i} = 0] = \delta_{00}, \quad \mathbb{E}[Y_i \mid T_{1i} = 0, T_{2i} = 1] = \delta_{01},$$
$$\mathbb{E}[Y_i \mid T_{1i} = 1, T_{2i} = 0] = \delta_{10}, \quad \mathbb{E}[Y_i \mid T_{1i} = 1, T_{2i} = 1] = \delta_{11}$$
- Under randomization of (T_{1i}, T_{2i}), parameters are causal: they correspond to the expectation of potential outcomes
- CEF is linear
$$\mathbb{E}[Y_i \mid T_{1i}, T_{2i}] = \alpha + \beta_1 T_{1i} + \beta_2 T_{2i} + \beta_3 T_{1i} T_{2i}$$
where
$$\alpha = \delta_{00}, \beta_1 = \delta_{10} - \delta_{00}, \ \beta_2 = \delta_{01} - \delta_{00}, \ \text{and}$$
$$\beta_3 = \delta_{11} - \delta_{10} - \delta_{01} + \delta_{00}$$

Interpretation of Interaction Terms

- OLS estimator identifies $(a, \beta_1, \beta_2, \beta_3)$:
$$(a, \beta_1, \beta_2, \beta_3) = \underset{a,b_1,b_2,b_3}{\operatorname{argmin}} \mathbb{E}\left[(Y_i - a - b_1 T_{1i} - b_2 T_{2i} - b_3 T_{1i} T_{2i})^2\right]$$
$$(\hat{a}, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3) \equiv \underset{a,b_1,b_2,b_3}{\operatorname{argmin}} \sum_{i=1}^{n} (Y_i - a - b_1 T_{1i} - b_2 T_{2i} - b_3 T_{1i} T_{2i})^2$$
- β_3 is an AIE: $\beta_3 = \delta_{11} - \delta_{10} - \delta_{01} + \delta_{00} = \xi\left((1,1)^\top; (0,0)^\top\right)$
- β_1 and β_2 are often called "main effects," but...
$$\beta_1 = \delta_{10} - \delta_{00} = \mathbb{E}[Y_i(1,0)] - \mathbb{E}[Y_i(0,0)]$$
$$\beta_2 = \delta_{01} - \delta_{00} = \mathbb{E}[Y_i(0,1)] - \mathbb{E}[Y_i(0,0)]$$
they are *conditional* effects when the other treatment takes 0
- Estimation of some ACEs requires addition of parameters, e.g.,
$$\widehat{\tau\left(\begin{pmatrix}1\\1\end{pmatrix}; \begin{pmatrix}1\\0\end{pmatrix}\right)} = \hat{\beta}_2 + \hat{\beta}_3$$
$\rightsquigarrow$ need inference on joint sampling distribution of parameters: multiple regression

Summary of First Five Weeks

- Potential outcomes framework for causal inference:
  1. Treatment effect is defined by difference b/w potential outcomes
  2. Only one potential outcome per unit is observed
  3. Average, not individual, treatment effects are of interest
- Randomized experiments:
  1. Randomization $\Longrightarrow$ potential outcomes $\perp\!\!\!\perp$ treatment $\Longrightarrow$ mean observed outcomes = mean potential outcomes
  2. Variance of potential outcome for each treatment condition
- Linear regression for experimental data:
  1. Linearity as consequence of design, not assumption
  2. Correspondence between causal and regression parameters
  3. Variance estimation from design-based perspective
- Some advanced topics I could not cover:
  1. Permutation tests and Fisher's sharp null
  2. Stratified/pairwise randomized experiments
  3. Average marginal interaction effects